SUB-OPTIMAL GAIN SCHEDULES FOR THE DISCRETE KALMAN FILTER

by

Robert Allen Crotteau

United States Naval Postgraduate School

THESIS

SUB-OPTIMAL GAIN SCHEDULES

FOR THE

DISCRETE KALMAN FILTER

by

Robert Allen Crotteau

June 1969

Sub-Optimal Gain Schedules for the Discrete Kalman Filter

by

Robert Allen Crotteau
Lieutenant (junior grade), United States Navy
B.S., United States Naval Academy, 1968

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
June 1969

ABSTRACT

The object of this study is to find an approximation to the discrete optimal Kalman filter gain schedule by closed-form analytic expressions. In doing so, required table storage and/or on-line computation time can be reduced at little expense in terms of filter performance degradation. The method of least squares was used to determine the closed-form solution which was the best fit to the discrete Kalman filter gain schedule. The criterion for performance degradation was the difference between the values of the diagonal elements of the estimation covariance matrix, $P_{k/k}$, obtained by using the Kalman gain schedule, and the corresponding values obtained by using the closed-form analytic expressions for the elements of the gain matrix. Examples are presented to show that near-optimal results were obtained utilizing this method. A comparison of the results of this study with another near-optimal estimation scheme is also included.

TABLE OF CONTENTS

I. INTRODUCTION - - - - - 7

II. THEORY - - - - - 9

III. PROCEDURE - - - - - 11

IV. RESULTS - - - - - 18

V. COMPARISON OF RESULTS WITH A PREVIOUS STUDY - - - - - 22

A. THE DISCRETE LINEAR PROBLEM - - - - - 22

VI. CONCLUSIONS - - - - - 27

VII. SUMMARY - - - - - 31

APPENDIX A Method of least squares - - - - - 32

APPENDIX B Sub-optimal gain equations - - - - - 34

BIBLIOGRAPHY - - - - - 35

INITIAL DISTRIBUTION LIST - - - - - 36

FORM DD 1473 - - - - - 37

TABLE OF SYMBOLS AND ABBREVIATIONS

SYMBOL	DIMENSION	NAME
$\underline{x}$	(n×1) vector	System state variables
A	(n×n) matrix	System matrix
B	(n×p) matrix	Distribution matrix
$\underline{u}$	(p×1) vector	Deterministic forcing functions
$\underline{w}$	(p×1) vector	Random forcing functions
$\underline{y}$	(m×1) vector	System outputs
H	(m×n) matrix	Measurement matrix
$\underline{z}$	(m×1) vector	System observables
$\underline{v}$	(m×1) vector	Random measurement noise signals
Φ	(n×n) matrix	State transition matrix
$P_{k/k}$	(n×n) matrix	Estimation covariance matrix
$P_{k/k-1}$	(n×n) matrix	Prediction covarianc3 matrix
G_k	(n×m) matrix	Gain matrix
Q	(n×n) matrix	Covariance of perturbation matrix
R	(n×m) scalar	Measurement noise covariance
N	(p×p) scalar	Random force covariance
I	(n×n) matrix	Identity matrix

LAPLACE DEFINITIONS

$$\Phi(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$$

$$\Gamma(t) = \mathcal{L}^{-1}\left[\frac{\Phi(s) + B}{s}\right]$$

I. INTRODUCTION

The theory of optimal estimation has undergone considerable investigation in recent years [1,2,9,11,12] . The objective of estimation is the production, in some optimal way, of estimates of the state vector, which is denoted by $\underline{x}_k$, from some set of observations, denoted by $\underline{z}_k$. In the early 1960's, R. E. Kalman initiated a new formulation of the Wiener filter theory expressing the results in the time domain rather than the frequency domain [6,7,8] .

The vector dynamic case of the linear filtering equation can be stated by

$$\underline{\hat{x}}_{k/k} = \underline{\hat{x}}_{k/k-1} + G_k \left[\underline{z}_k - \underline{z}_{k/k-1}\right] \tag{1}$$

where $\underline{\hat{x}}_{k/k}$ is the estimate of $\underline{x}_k$. Kalman found that the optimal estimate $\underline{\hat{x}}_{k/k}$, in a minimum-variance sense, could be obtained by solving the difference equations

$$G_k = P_{k/k-1}H^T\left[HP_{k/k-1}H^T + R\right]^{-1} \tag{2}$$

$$P_{k/k} = (I-GH)P_{k/k-1} \tag{3}$$

$$P_{k+1/k} = \Phi P_{k/k}\Phi^T + Q \tag{4}$$

with $P_{0/-1} - \mathrm{VAR}\left[\underline{\hat{x}}_0 - \underline{x}_0\right]$ and $\underline{\hat{x}}_0 = E\left[\underline{x}_0\right]$. The definitions and dimensions of the matrix quantities in the above equations may be found in the Table of Symbols and Abbreviations.

It is obvious that, in order to solve for successive values of $\underline{\hat{x}}_{k/k}$ in equation (1), it is necessary to solve equations (2), (3), and (4) recursively. This process involves a considerable number of matrix manipulations. Consequently the on-line computation time would be large. Also considerable storage space would be needed if the problem were to be redone at a later time, utilizing the same matrix of gains. This estimation scheme requires the implementation of $\frac{n + n(n+1)}{2}$ difference equations or an n^{th} order system of which $\frac{n(n+1)}{2}$ are non-linear. For large-order systems, this scheme may be of such complexity that it is undesirable for a given application.

If it were possible to derive an analytic closed-form expression which would approximate the optimal Kalman filter gain schedule with acceptable degradation of filter performance then on-line computation time would be greatly decreased since no matrix manipulations would be necessary. Also, the need for table storage of the gains, G_k, would not be necessary since it would be a simple matter to calculate them again when they were needed.

II. THEORY

Appendix A illustrates that, according to the criterion of least squares, the best fit curve to a set of data is one in which the sum of the squares of the errors of the points is a minimum. The method used to obtain this best fit curve is known as the Method of Least Squares.

If this method can be used to produce a curve which is the best fit, in a least-squares sense, to the set of optimal gains, then the equation for that curve would be a sub-optimal closed-form solution for those gains.

Therefore, all that is necessary to produce a candidate closed-form analytical approximation for the optimal Kalman filter gain schedule is to fit a least squares curve to the set of optimal gains.

Observations of typical optimal Kalman filter gain schedules indicated that these gains would most likely be best fit with exponential curves. In order to take advantage of existing computer programs on least-squares curve-fitting it is necessary to convert the exponential form to a linear form. Appendix A illustrates the method by which this transformation was made. Once the transformation to a linear form has been made the method of least-squares curve-fitting can be applied and the results, through an inverse transformation, can be used to obtain the desired exponential expressions.

Equations (2), (3), and (4) reveal that the solution for the optimal gain matrix depends on the quantities $P_{k/k-1}$, H, R, Φ and Q where

$$Q = \Gamma N \Gamma^T \tag{5}$$

and N is the random force covariance associated with the particular problem. The definitions and dimensions of these quantities may be found in the Table of Symbols and Abbreviations. In order to reduce the complexity of the system, for that is the basic goal of this study, some simplifying assumptions are made. It is necessary to initialize the prediction covariance matrix $P_{k/k-1}$. The diagonal elements of this matrix indicate the confidence given to the initial filter state vector. It will be assumed that little confidence is given, thus the diagonal elements of $P_{0/-1}$ will be large. The off-diagonal elements indicate to what extent the initialization errors of the filter states are dependent. It will be assumed that they are independent and thus the off-diagonal terms of $P_{0/-1}$ will be zero.

It is also assumed that the measurement matrix, H, the state transmission matrix, Φ, and the Γ matrix, which is used to compute the covariance of perturbation matrix, Q, are constant for a given plant. That is to say that once these quantities are known for a specific process or plant, they remain unchanged during the operation of the plant or process.

Thus, once the nature of the plant or process is known, the only variables involved in the determination of the optimal gain matrix are the measurement noise variance, R, and the random force variance, N. These quantities will be assumed constant in time for a given problem, but may vary from problem to problem.

With these assumptions it is possible to develop closed-form expressions for the elements of the gain matrix in terms of the quantities R and N.

III. PROCEDURE

As stated above the object of this study is to develop a method for finding a sub-optimal closed-form analytic expression for the optimal Kalman filter gain schedule which would be easier to implement and reduce required table storage and/or on-line computation time while keeping filter degradation to a minimum.

The best way to present the method of this study is to follow an illustrative example which demonstrates the technique used to obtain the desired results. Note that this method is designed to be used with a plant or process which has a known, non-variable configuration, with the only variables being the measurement noise variance, R, and the random force variance, N.

The problem discussed for purposes of demonstration will be one where the motion of a mass particle in one dimension is considered. The observer will initialize the problem by indicating when he sees the particle pass a zero point. Subsequent measurements of the particle's position will be made at a period of T seconds as measured by a clock. The filter then must produce minimum variance estimates of the particle's position and velocity at discrete times kT where $k \geq 0$.

The system state variables chosen were position and velocity. The vector matrix form of the state equations can be expressed as

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} f \quad (6)$$

Assuming that the random forcing function, f, is piecewise constant, and recalling the definitions of $\Phi(T)$ and $\Gamma(T)$ given in the Table of Symbols and Abbreviations, the solution of equation (6) is given by

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Gamma \left[\underline{u}_k + \underline{w}_k\right] \tag{7}$$

where

$$\Phi(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{8}$$

and

$$\Gamma(T) = 1/M \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} . \tag{9}$$

Since only measurements of position are made the measurement matrix is given by

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} . \tag{10}$$

From equation (5) the covariance of perturbation matrix, Q, is given by

$$Q = N/_{M^2} \begin{bmatrix} \frac{T^4}{4} & \frac{T^3}{2} \\ \frac{T^3}{2} & T^2 \end{bmatrix} . \tag{11}$$

Next the problem must be initialized. Assuming that the observer signals when he thinks that the particle is initially at a position zero and also assuming his guess of the particle's velocity to be 10.0 units/sec., the filter's initial state vector is

$$\hat{\underline{x}}_{0/-1} = \begin{bmatrix} 0.0 \\ 10.0 \end{bmatrix} . \tag{12}$$

As discussed above, when initializing the prediction covariance matrix it is necessary to consider how much confidence is given to the values of the observer's eye-sight and reflexes as well as his ability to estimate the particle's initial velocity. Being distrustful of his ability the initial covariance matrix becomes

$$P_{0/-1} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \tag{13}$$

The diagonal elements indicate the lack of confidence in the initialization of the state vector of equation (12).

Assuming a value of unity for the mass of the particle, values of R and N may be chosen and the optimal gain schedule for the Kalman filter may be found using equations (2) through (4).

The problem to be considered now is how to achieve a closed-form analytic approximation for G_k so that equation (1) may be solved to obtain estimates of $\underline{x}_k$.

Before going further a quantity to be known as the pseudo signal-to-noise ratio of the system must be defined. This signal-to-noise ratio is defined as N/R where

$$N = E\left[\underline{w}_k\ \underline{w}_k^T\right] \tag{14}$$

and

$$R = E\left[\underline{v}_k\ \underline{v}_k^T\right] \quad . \tag{15}$$

To make this signal-to-noise ratio a scalar quantity it must be stipulated that the random measurement noise signal, $\underline{v}_k$, and the random forcing function, $\underline{w}_k$, are scalar quantities for this problem. Thus the quantity N/R is a scalar measurement of the signal-to-noise ratio of the system.

The example problem, with initial conditions as shown above, was run for various values of the signal-to-noise ratio N/R. Fig. 1 and Fig. 2 show a family of curves for the gain matrix elements G(1,1) and G(2,1) repsectively and the associated signal-to-noise ratio of each. As stated earlier it was decided that the points of G(1,1) could best be approximated by the sum or difference of two exponentials. Using the method illustrated in Appendix A of reducing an exponential to linear form it was a simple matter to take the logarithms of the points and obtain a least-squares fit curve for each set of points. The curves obtained as well as the equations for these curves are found in Fig. 1 and Fig. 2.

Steady-state gain as a function of signal-to-noise ratio is plotted in Fig. 3. These plots indicate that the steady-state gains also behave in an exponential manner with respect to the signal-to-noise ratio. Further investigation showed that the exponents of the equations in Fig. 1 and Fig. 2 behave in a like manner. Using the same method as discussed above, closed-form solutions for these quantities were developed in terms of the signal-to-noise ratio N/R. The analytical closed-form solution and its related equations are called the sub-optimal equations. These appear in equations (4.1) through (4.9) of Appendix B.

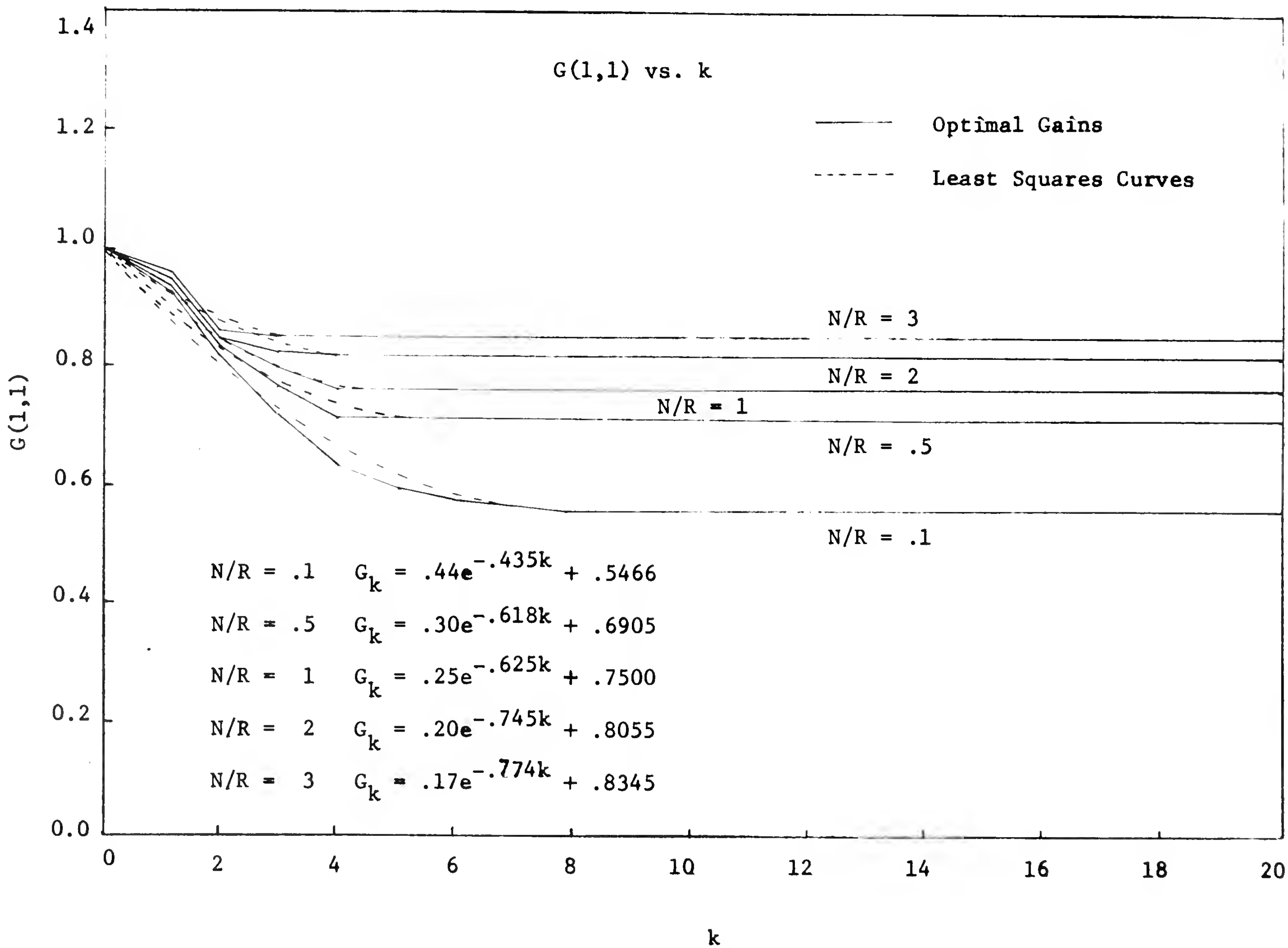
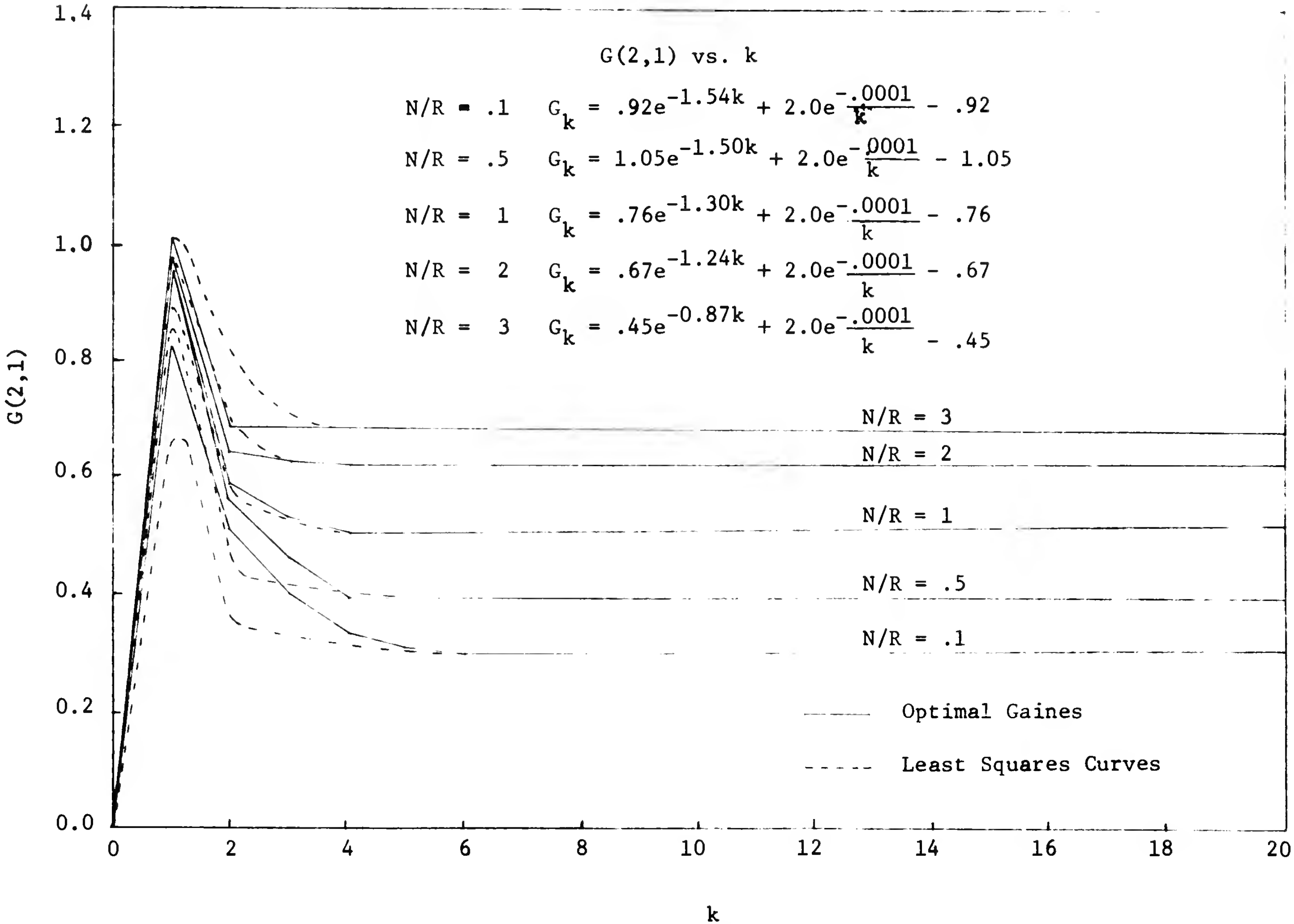

G(1,1) vs. k
Optimal Gains
Least Squares Curves
N/R = 3
N/R = 2
N/R = 1
N/R = .5
N/R = .1
N/R = .1 $G_k = .44e^{-.435k} + .5466$
N/R = .5 $G_k = .30e^{-.618k} + .6905$
N/R = 1 $G_k = .25e^{-.625k} + .7500$
N/R = 2 $G_k = .20e^{-.745k} + .8055$
N/R = 3 $G_k = .17e^{-.774k} + .8345$
G(1,1)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
k

Fig. 1



G(2,1) vs. k
N/R = .1 $G_k = .92e^{-1.54k} + 2.0e^{-\frac{.0001}{k}} - .92$
N/R = .5 $G_k = 1.05e^{-1.50k} + 2.0e^{-\frac{.0001}{k}} - 1.05$
N/R = 1 $G_k = .76e^{-1.30k} + 2.0e^{-\frac{.0001}{k}} - .76$
N/R = 2 $G_k = .67e^{-1.24k} + 2.0e^{-\frac{.0001}{k}} - .67$
N/R = 3 $G_k = .45e^{-0.87k} + 2.0e^{-\frac{.0001}{k}} - .45$
N/R = 3
N/R = 2
N/R = 1
N/R = .5
N/R = .1
Optimal Gaines
Least Squares Curves
G(2,1)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
k

Fig. 2

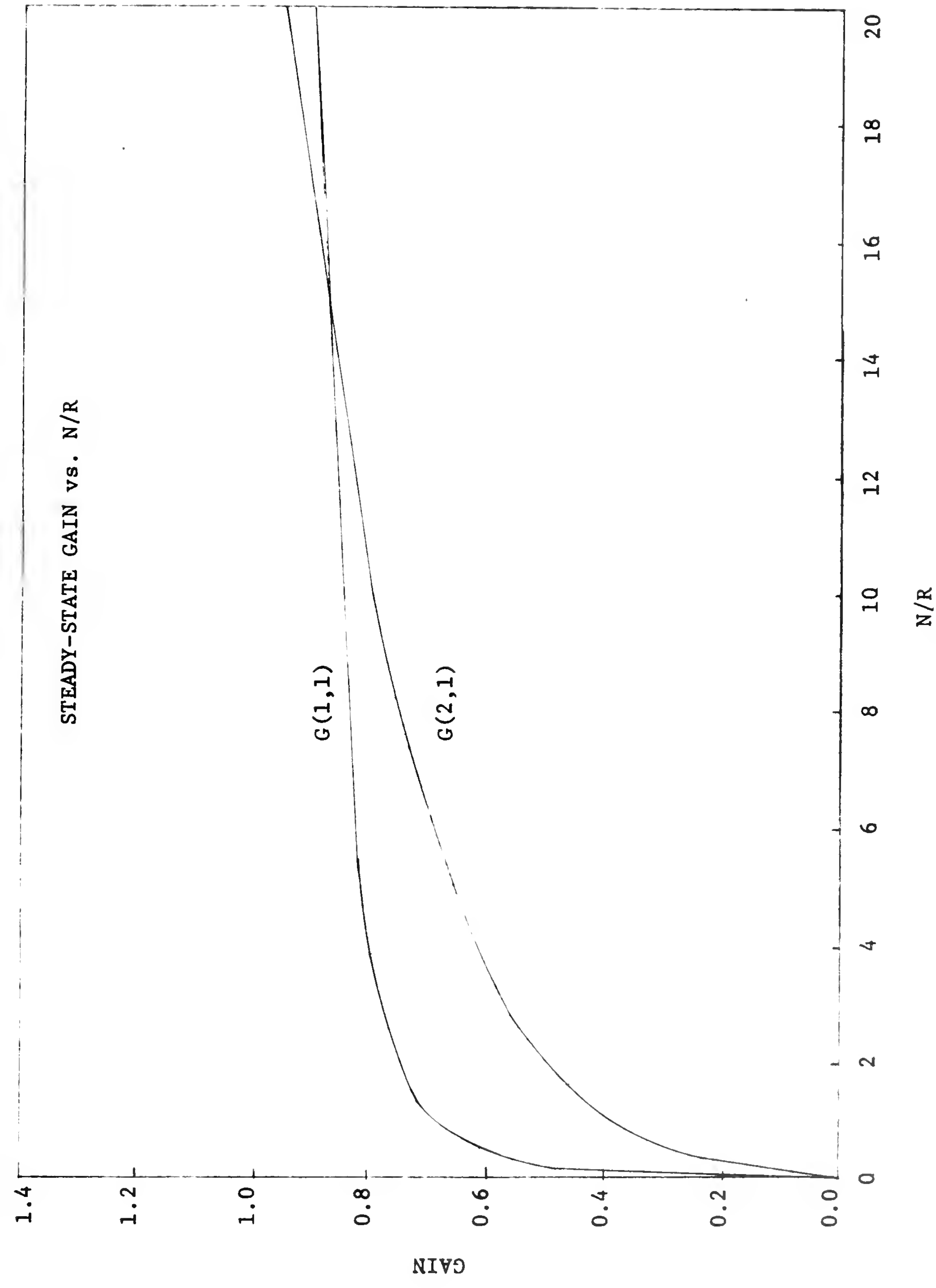

STEADY-STATE GAIN vs. N/R
G(1,1)
G(2,1)
GAIN
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16
18
20
N/R

Fig. 3

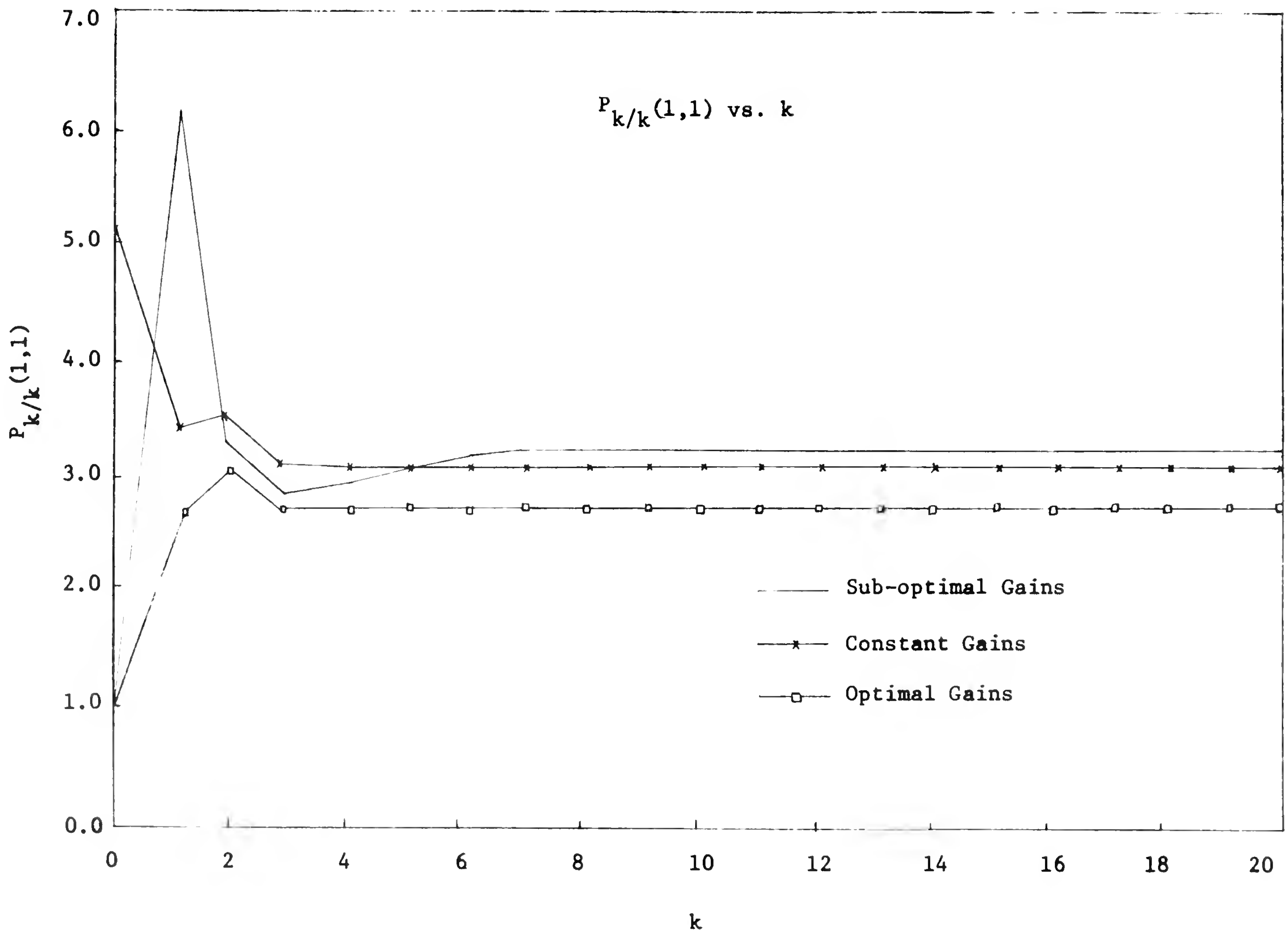
IV. RESULTS

Once the closed-form analytic approximation for the gain matrix, G_k, is found it is an easy matter to test the results against the recursive equations of the optimal Kalman filter. Since the Kalman filter gains are optimum in a minimum-variance sense it is necessary only to compare the diagonal elements of the $P_{k/k}$ matrix obtained using the closed-form solution for G_k with the respective elements of the $P_{k/k}$ matrix obtained using the optimal Kalman filter gain schedule.

Since the gains obtained from the solution of the closed-form equations are non-optimal, equation (3) may not be used. Instead it is necessary to derive a general equation for all values of $P_{k/k}$ obtained with other than optimal gains. Such an equation has been derived[4] which can be expressed as

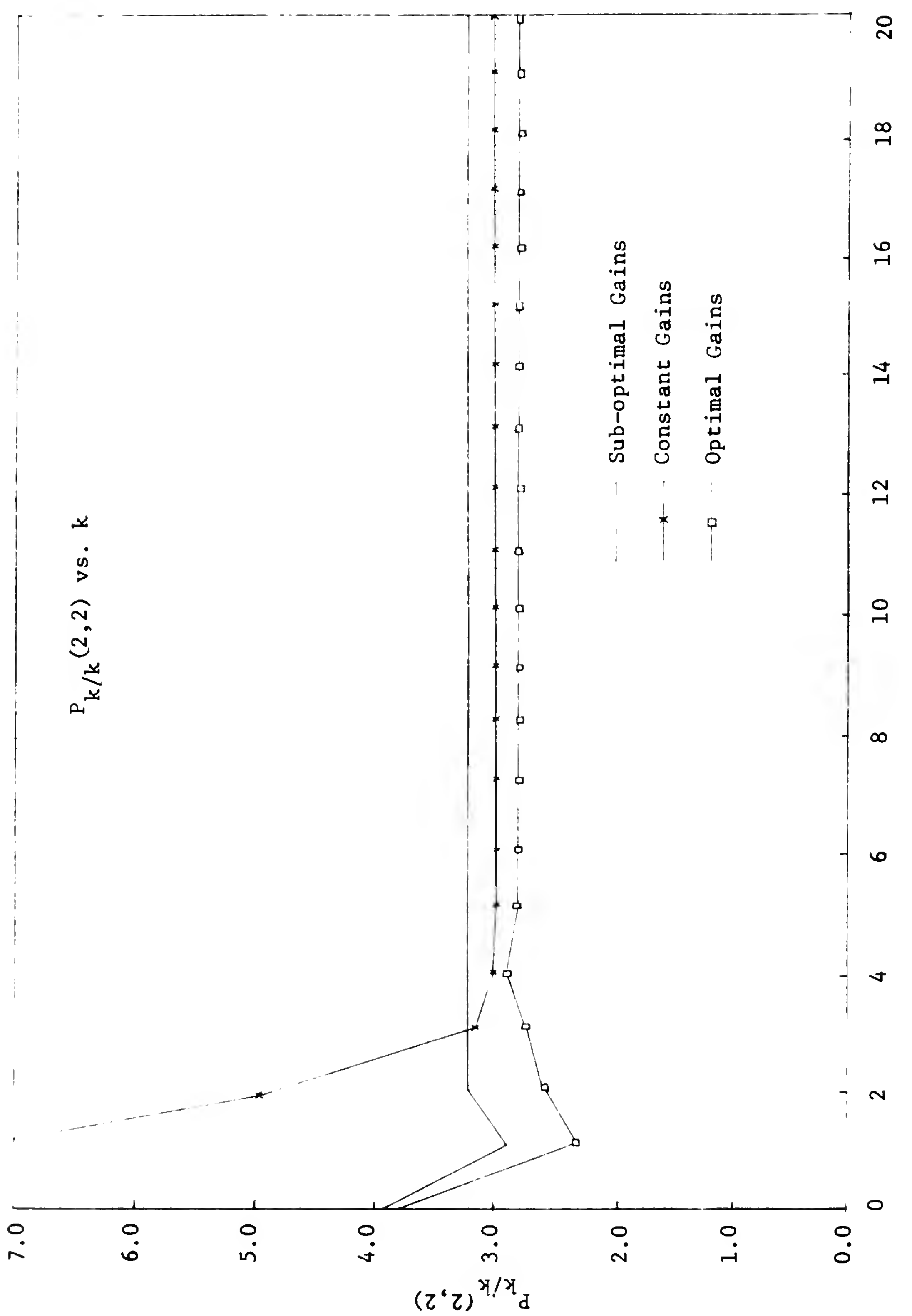
$$P_{k/k} = P_{k/k-1} - G_k H P_{k/k-1} H^T G_k^T + G_k (H P_{k/k-1} H^T + R) G_k^T \; . \tag{16}$$

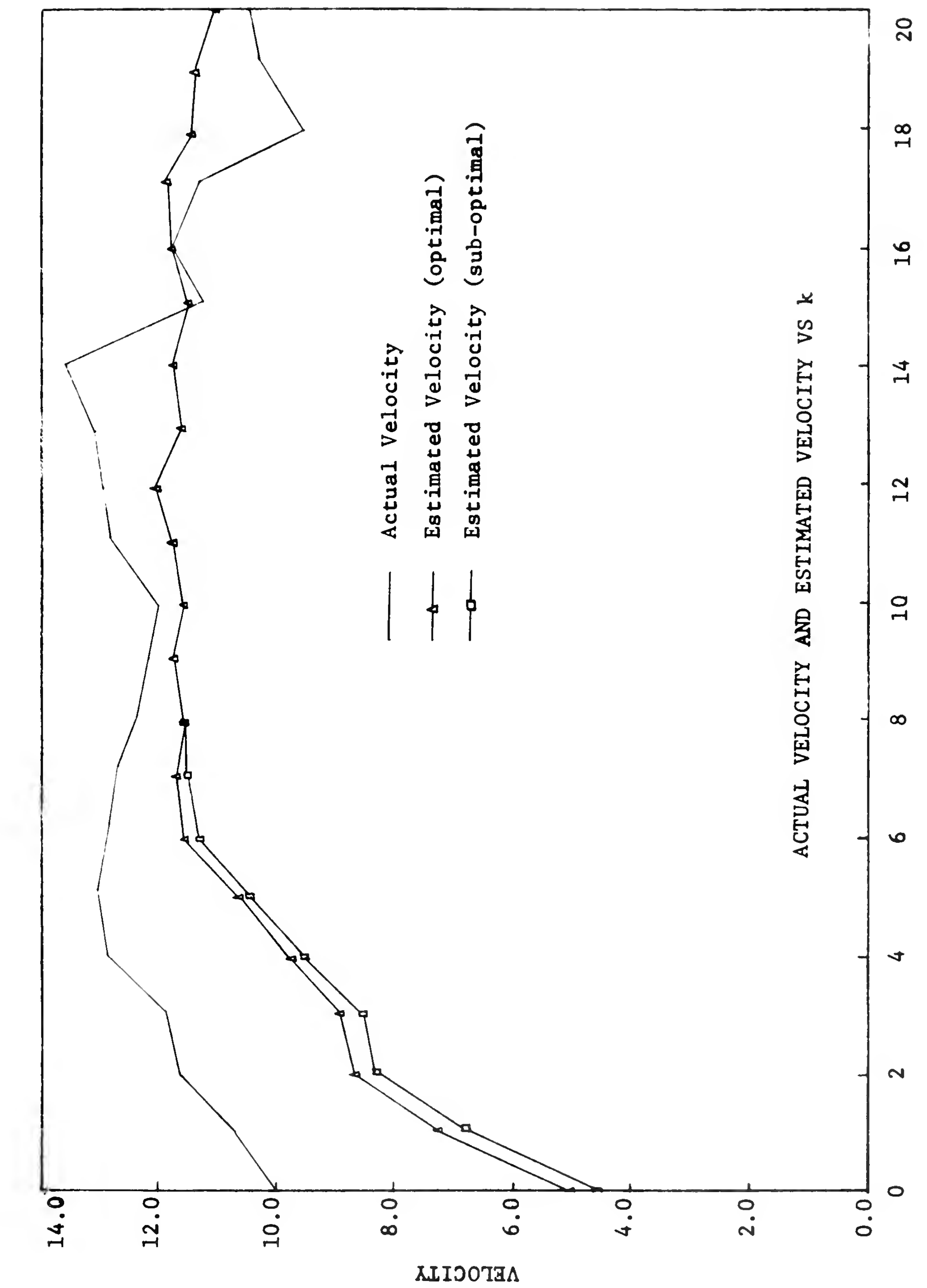
Figures 4 and 5 show the comparison of the diagonal elements of $P_{k/k}$ for the optimal case and those for the case where G_k was determined by the closed-form expressions developed. These figures indicate that the values obtained using sub-optimal gains are higher than the optimal values. To determine how much this difference affects the estimation capabilities of the filter this problem was simulated and a comparison was made between the actual value of velocity for the point and the estimated value of velocity for both the optimal and sub-optimal case. Figure 6 indicates that, for this problem, the estimating capabilities of the optimal and sub-optimal filters are similarly effective.



$P_{k/k}(1,1)$ vs. k
Sub-optimal Gains
Constant Gains
Optimal Gains
$P_{k/k}(1,1)$
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
0
2
4
6
8
10
12
14
16
18
20
k

Fig. 4


$P_{k/k}(2,2)$ vs. k
Sub-optimal Gains
Constant Gains
Optimal Gains
$P_{k/k}(2,2)$
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
0
2
4
6
8
10
12
14
16
18
20
k
Fig. 5



ACTUAL VELOCITY AND ESTIMATED VELOCITY VS k

Fig. 6

V. COMPARISON OF RESULTS WITH A PREVIOUS STUDY

Sims and Melsa[13] have reported on attempts to achieve, for the continuous case, near-optimal estimation using a structure which is easier to implement than the optimal Kalman solution. Their method, called Specific Optimal Estimation, is described in Ref. 13. The following is an attempt to adapt their methods and results for the continuous case to the discrete case.

A. THE DISCRETE LINEAR PROBLEM

Consider the linear message model

$$\underline{x}_{k+1} = \Phi \underline{x}_k + \Gamma \underline{u}_k \tag{17}$$

with observation model

$$\underline{z}_k = H\underline{x}_k + \underline{v}_k \tag{18}$$

where u and v are white noise processes such that

$$\begin{aligned} E\left[\underline{u}_k \; \underline{u}_k^T\right] &= Q \\ E\left[\underline{v}_k \; \underline{v}_k^T\right] &= R \\ E\left[\underline{u}_k \; \underline{v}_k\right] &= 0 \\ E\left[\underline{u}_k\right] = E\left[\underline{v}_k\right] &= 0 \end{aligned} \tag{19}$$

The optimal linear minimum variance estimate, $\hat{\underline{x}}_{k/k}$, of the state vector $\underline{x}_k$ is given by

$$\hat{\underline{x}}_{k/k} = \hat{\underline{x}}_{k/k-1} + G_k \left[\underline{z}_k - \underline{z}_{k/k-1}\right] \tag{20}$$

where

$$G_k = P_{k/k-1}H^T\left[HP_{k/k-1}H^T + R\right]^{-1} \quad (21)$$

and $P_{k/k}$ is the solution to the matrix difference equations

$$P_{k/k} = (I-GH)P_{k/k-1} \quad (22)$$

$$P_{k+1/k} = \Phi P_{k/k}\Phi^T + Q \quad (23)$$

and $P_{0/-1} = \text{VAR}\left[\hat{\underline{x}}_0 - \underline{x}_0\right]$; $\hat{\underline{x}}_0 = E\left[\underline{x}_0\right]$.

It is proposed in this paper, to find an estimate $\hat{\underline{x}}_{k/k}$ of $\underline{x}_k$ having the same form as equation (20), but with the gain matrix, G_k, constrained to be of a specific configuration. In particular, G_k is assumed to be the solution to the difference equation

$$G_k = F(k,\underline{a}) \qquad G_0 = D \quad (24)$$

where $\underline{a}$ is a constant-parameter vector to be determined optimally and D is a matrix of initial conditions which may or may not be determined optimally.

Let $\tilde{\underline{x}}_k = \underline{x}_k - \hat{\underline{x}}_{k/k}$ be the estimation error, and assume that the observation interval be discrete. The error criterion used is

$$J = \sum_{k=1}^{T} E\left[\|\tilde{\underline{x}}\|^2\right] . \quad (25)$$

This may be written as

$$J = \text{tr} \sum_{k=1}^{T} E\left[\tilde{\underline{x}}\,\tilde{\underline{x}}^T\right] . \quad (26)$$

Making use of the definition

$$P_{k/k} = E\left[\tilde{\underline{x}}\,\tilde{\underline{x}}^T\right] \quad (27)$$

equation (26) can be written

$$J = \mathrm{tr} \sum_{k=1}^{T} P_{k/k} \quad . \tag{28}$$

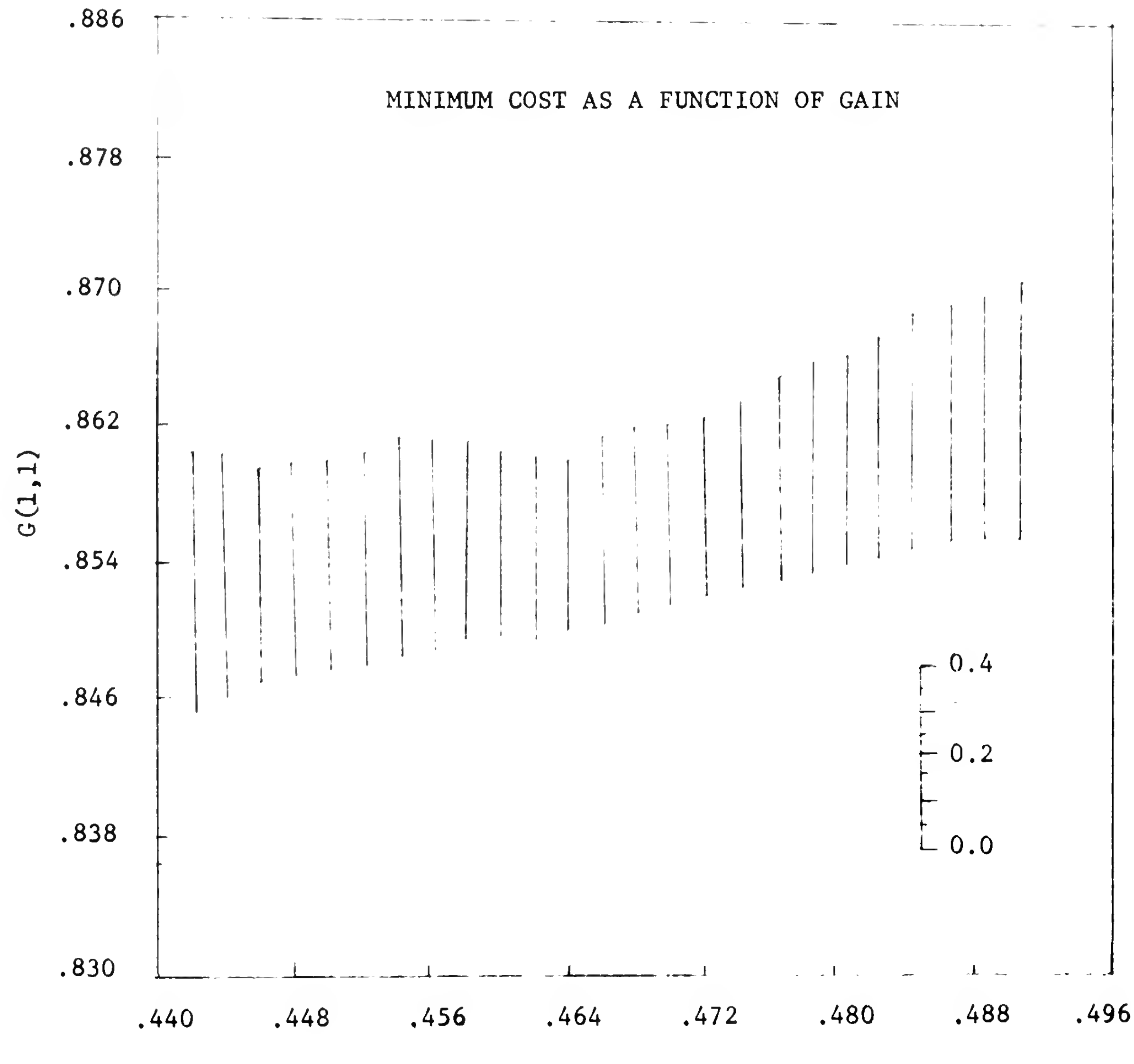
The problem then is to determine the parameters in the equation of G_k such that J is minimized. If it is assumed that the estimate, $\underline{\hat{x}}_{k/k}$, is of the form specified in equation (20) it is easy to verify that $P_{k/k}$ satisfies the difference equation

$$P_{k/k} = P_{k/k-1} - G_k H P_{k/k-1} H^T G_k^T + G_k (H P_{k/k-1} H^T + R) G_k^T \quad . \tag{29}$$

Sims and Melsa determined the parameters in the equation for G_k which minimized equation (28) by developing the Hamiltonian for $P_{k/k}$, determining the equations for the Lagrange multiplier matrices, and obtaining the solution by solving the problem as a two-point boundary problem with appropriate boundary conditions.

Since the equation for $P_{k/k}$ is a difference equation, in the discrete case, rather than a differential equation, this approach cannot be used. One might solve equation (29) recursively with equation (4) for all values of k in the summation, and set the derivative with respect to the undetermined parameters of G_k equal to zero to determine the gains which would minimize equation (28), but an inspection of this method by the author revealed that this would be too tedious. The approach adopted here, although being equally tedious, lent itself to a computer solution. This approach involved setting the gain matrix to a constant in equation (29) and evaluating the cost function of equation (28). One element of the gain matrix was held constant and the other was varied until a minimum value for J was found. The element of the gain matrix which had been held constant was then incremented and the procedure was duplicated. As a result Fig. 9 was obtained. The verticle lines are proportional to the value of the cost function at each local minimum shown. The coordinates of the base of each verticle line are the values of G(1,1) and G(2,1) which gave the value of the cost function shown. The values of gain at which the cost function reached its absolute minimum were G(1,1) = .852 and G(2,1) = .466. These are compared with the optimal steady-state values of G(1,1) = .700 and G(2,1) = .400. Using the values of gain obtained above the illustrative problem was run and the results are presented in graphic form in Fig. 4 and Fig. 5.

Note that the gains obtained above were close to the steady-state values of the optimal gain matrix. It would be expected that these gains would become closer to the optimal steady-state values as the number of samples increases. To show this the solution was performed again for ten samples, vice twenty. The values of the gain matrix for minimum cost obtained were G(1,1) = .902 and G(2,1) = .516. Thus, with a smaller number of samples, the sub-optimal constant gains are further from the steady-state optimal gains.


MINIMUM COST AS A FUNCTION OF GAIN
.886
.878
.870
.862
.854
.846
.838
.830
G(1,1)
.440
.448
.456
.464
.472
.480
.488
.496
0.4
0.2
0.0
G(2,1)
Fig. 9

VI. CONCLUSIONS

It is evident from Fig. 4 through Fig. 6 that the Kalman-type filter utilizing the closed-form equations for sub-optimal gains performs in a near-optimal manner with little degradation in the performance of the filter. The question now arises as to whether this method of determining the filter gains does fulfill the objective of this study; that is, has required table storage and/or on-line computation time been reduced? A comparison of equations (4.1) and (4.2) of Appendix B with equation (2) above indicates that the former equations are easily and quickly solved for all values of k, once the signal-to-noise ratio, N/R, is determined, while in the latter equation it is necessary to perform four matrix products, two transposes, and a matrix inversion in order to solve for one value of the gain matrix. In order to solve this equation for all values of k it is necessary to solve equations (3) and (4) recursively along with equation (2). This procedure requires four additional matrix products, an additional matrix transpose, and a matrix addition and subtraction in order to obtain each additional value for the gain matrix. Thus the closed-form gain equations will greatly reduce on-line computation time.

This same argument can be used to show that required table storage has also been reduced. Assume, for instance, that a problem is to be performed over again at a later time. In the case of the Kalman filter it is necessary to store the gain schedule, since it would be too costly in on-line computation time to re-compute the schedule. Since the closed-form equations are much more quickly solved it is not necessary to store the values of gain obtained in this manner.

Recall that it has been assumed that all factors which influence the gain matrix are to be constant from problem to problem, with the exception of the scalar measurement-noise covariance, R, and the scalar random-force covariance, N. These quantities were assumed constant for a single problem but could change from problem to problem. Suppose now that one or both of these quantities changed during the problem, say at time k equal to ten. The optimal gain schedule for such a system might be as shown in Fig. 7. In a situation such as this the Kalman gain schedule would have to be computed on-line to maintain optimality if the change time were not known a priori. In Fig. 8 a plot is shown of the reaction of the closed-form approximations to such a situation. Although the sub-optimal gains are still non-optimal, they follow the general trend of the optimal gains, and hence may be used in their stead.

The arguments presented above would indicate that substantial savings in time and space could be realized at little cost in performance if the closed-form approximations for the gain matrix were used in place of the Kalman recursive solution.

An inspection of Fig. 4 and Fig. 5 reveals that the results obtained by the specific optimal estimation technique offered by Sims and Melsa are more near-optimal than those obtained by the method discussed in this paper. This fact brings out the major difference between these two methods. In this study the gain matrix was constrained to be an exponential, the parameters of which were determined by geometric means. That is, the parameters of the exponential were determined by requiring that the exponential be a geometric least-squares fit to the Kalman filter optimal-gain matrix. In contrast the parameters of the constant-gain matrix of the specific optimal estimator were determined by minimizing errors in filter performance. It is significant to note that the constant gains employed in the Sims/Melsa method gave better performance than the exponential gains of the geometric fit method. One might speculate that the assumption of an exponential form in the former method might even more closely match the optimal filter performance.

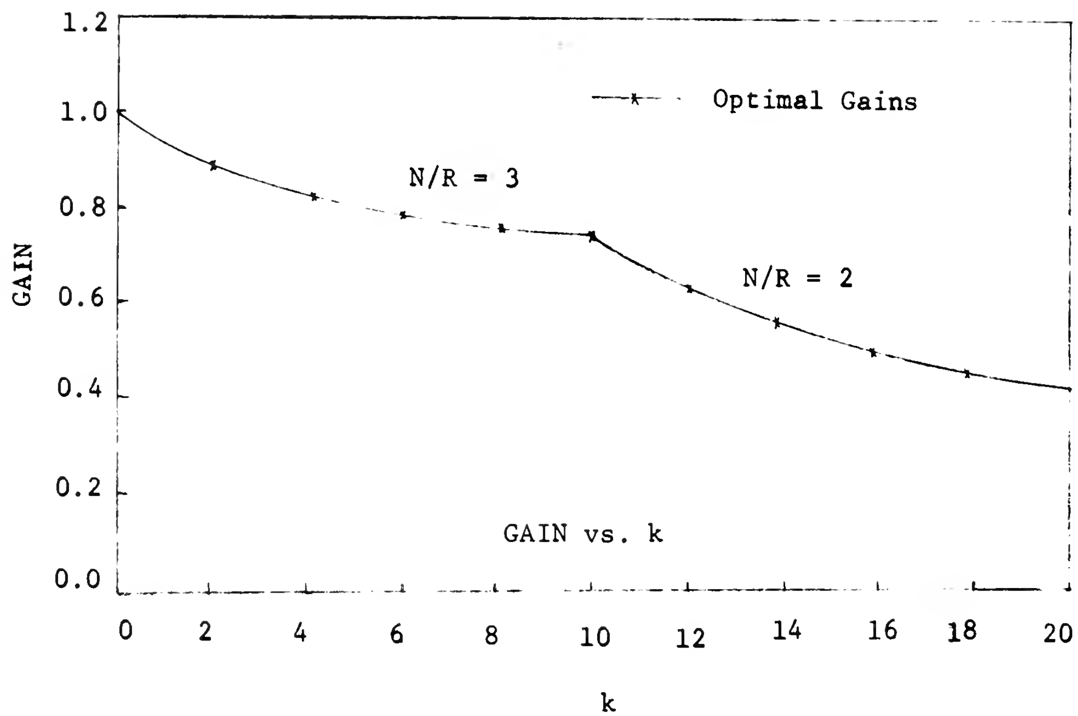

1.2
1.0
0.8
0.6
0.4
0.2
0.0
GAIN
Optimal Gains
N/R = 3
N/R = 2
GAIN vs. k
0
2
4
6
8
10
12
14
16
18
20
k

Fig. 7

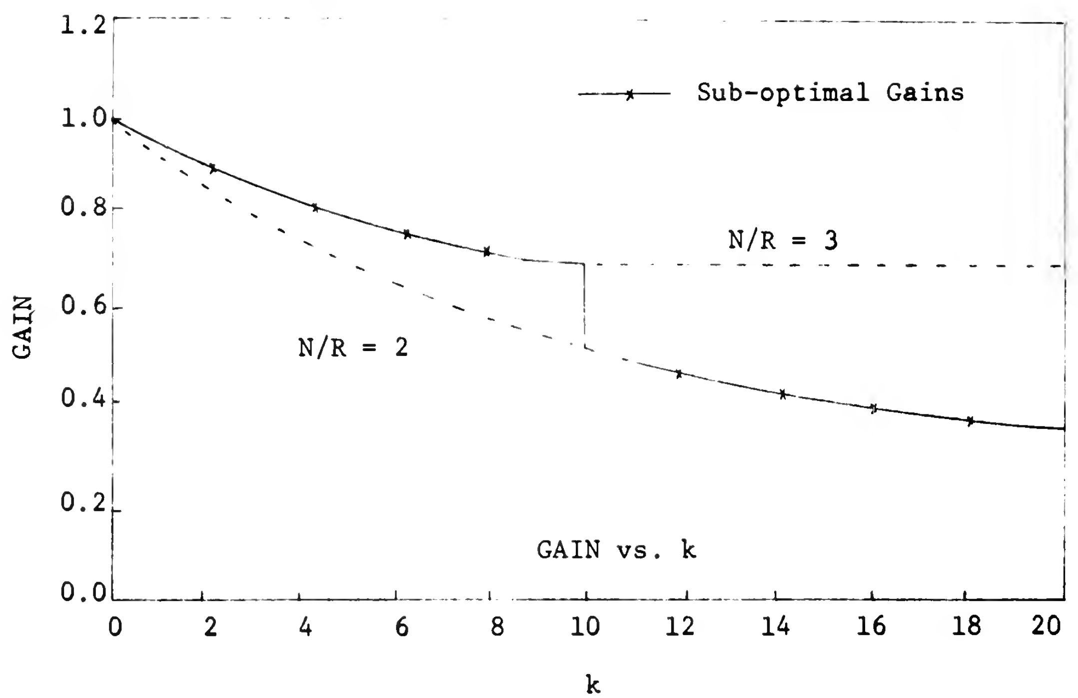

1.2
1.0
0.8
0.6
0.4
0.2
0.0
GAIN
Sub-optimal Gains
N/R = 3
N/R = 2
GAIN vs. k
0
2
4
6
8
10
12
14
16
18
20
k

Fig. 8

VII. SUMMARY

This study has illustrated that a closed-form analytic approximation to the filter gain matrix for an optimal Kalman filter can be obtained which will reduce the computation time while procuding near-optimal estimates. This method would be useful in problems where computation time and storage space were limited. This method has compared favorably with both the optimal filter and the specific optimal estimator with which it was compared.

APPENDIX A

METHOD OF LEAST SQUARES

The most probable value of a quantity which is obtained from a set of observations is the one which corresponds to the most probable set of errors of observation. These errors are known as residuals. Consider a set of n observations of equal precision, the mose probable errors of which are x_1, x_2, x_3,...x_n, respectively. Since the probability of the simultaneous occurrence of several events in a series is the product of their individual probabilities, and the probability of an error, x, is

$$p = \frac{h}{\sqrt{\pi}} \; e^{-h^2x^2} \tag{2.1}$$

it follows that the probability of the simultaneous occurrence of the errors x_1, x_2, x_3,...x_n will be

$$P = \left[\frac{h}{\sqrt{\pi}} \; e^{-h^2x_1^2}\right]\left[\frac{h}{\sqrt{\pi}} \; e^{-h^2x_2^2}\right] \cdots \left[\frac{h}{\sqrt{\pi}} \; e^{-h^2x_n^2}\right] ;$$

that is

$$P = \left[\frac{h}{\sqrt{\pi}}\right]^n \; e^{-h^2(x_1^2 \; x_2^2 \; x_3^2 \ldots x_n^2)} \; . \tag{2.2}$$

Since h, n, , and e are constants in a given problem the quantity P is a maximum when the exponent of e is a maximum, or when the sum

$$x_1^2 \; x_2^2 \; x_3^2 \ldots x_n^2 \quad \text{is a minimum.} \tag{2.3}$$

Thus, the most probably value of the observed quantity, or the best value, in other words, obtainable from the given set of observations, will be one for which the sum of the squares of the residuals is a minimum. This is called the Principle of Least Squares.

The Principle of Least Squares has been used extensively to formulate computer programs for determining the coefficients of a linear equation which is a best fit, in the Least-Square sense, to observed data. In general, equations of higher than first degree can be reduced to linear form by developing the function by Taylor's Theorem and neglecting the squares, products, and higher powers of the small increments involved.

Equations in which the unknown constant occurs as an exponent constitute a special case of reduction to linear form. In brief, the method is to throw the equation into the logarithmic form by taking the logarithm of each member. The resulting function will be linear with respect to the desired coefficient. Suppose the function is of the form

$$y = ae^{bt} \tag{2.4}$$

in which a and b are to be determined so as to fit all to the observations as well as possible. Taking the logarithm of each member gives

$$\log y = \log a + bt, \tag{2.5}$$

which is the linear form of

$$y' = A + bt, \tag{2.6}$$

where A and b are the unknown constants. By simply taking the inverse logarithm of the coefficient A the Least Squares fit curve of exponential form can be obtained.

APPENDIX B

SUB-OPTIMAL GAIN EQUATIONS

$$G(2,1)_k = Ae^{-\alpha k} + 2e^{-\frac{.0001}{k}} + A \qquad (4.1)$$

$$G(1,1)_k = Ce^{-\beta k} + D \qquad (4.2)$$

$$\text{ratio} = N/R \qquad (4.3)$$

$$E = e^{-\frac{.4325}{\text{ratio}}} \qquad (4.4)$$

$$D = e^{-\frac{.6841}{\text{ratio}}} \qquad (4.5)$$

$$A = 2 - E \qquad (4.6)$$

$$C = 1 - D \qquad (4.7)$$

$$\alpha = 2e^{-.5383(\text{ratio})} \qquad (4.8)$$

$$\beta = e^{-\frac{.7382}{\text{ratio}}} \qquad (4.9)$$

BIBLIOGRAPHY

1. Agarwal, G. C. and R. Sridhar, "Design of Specific Optimal Control Systems," I.E.E.E. Region Six Conference Record, Vol. 2, 1966, pp. 717-726.

2. Bryson, A. E. Jr., and L. J. Henrikson, "Estimation Using Sampled Data Containing Sequentially Correlated Noise," Technical Report No. 533, Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass., 1967.

3. Coddington, E. F., and O. J. Marshall, Least Squares in Engineering, University of Toronto Press, Toronto, Canada, 1952.

4. Demetry, J. S., Notes for Statistical Control Theory, United States Naval Postgraduate School, Monterey, Calif., 1968.

5. Guest, P. P., Numerical Methods of Curve Fitting, University Press, Cambridge, Mass., 1961, pp. 370-381.

6. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," Journal of Basic Engineering, (ASME Trans.), 82D, 1960, pp. 35-45.

7. Kalman, R. E., and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," Journal of Basic Engineering (ASME Trans.), 83D, 1961, pp. 95-107.

8. Kalman, R. E., "New Methods in Wiener Filtering Theory," Proceedings of the First Symposium of Engineering Applications of Random Function Theory and Probability, Wiley, New York, 1963, pp. 270-389.

9. Lee, R.C.K., Optimal Estimation, Identification and Control, Research Monograph No. 28, M.I.T. Press, Cambridge, Mass., 1964.

10. Leland, O. M., Practical Least Squares, McGraw-Hill Book Company, New York, 1921, pp. 145-148.

11. Libelt, P. B., An Introduction to Optimal Estimation, Addison-Wesley Publishing Company, Menlo Park, Calif., 1967.

12. Neal, S. R., "Optimizing the Transient Performance of a Digital Filter-Predictor," Technical Note No. 304-118, U. S. Naval Ordnance Test Station, China Lake, Calif., 1965.

13. Sims, C. S., and J. L. Melsa, Specific Optimal Estimation, paper presented at Asilomar Conference on Control Theory, Pacific Grove, Calif., 1968.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	20
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Commander, Naval Ordnance Systems Command Headquarters Department of the Navy Washington, D. C. 20360	1
4.	Assoc. Professor J. S. Demetry, Code 52De Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	1
5.	LT(jg) Robert Allen Crotteau, USN 3865 Hunt Road Lapeer, Michigan 48446	1

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
Naval Postgraduate School Monterey, California 93940	Unclassified
	2b. GROUP

3. REPORT TITLE

Sub-Optimal Gain Schedules for the Discrete Kalman Filter

4. DESCRIPTIVE NOTES *(Type of report and, inclusive dates)*

Master's Thesis

5. AUTHOR(S) *(First name, middle initial, last name)*

Robert Allen Crotteau

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1969	36	13

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

~~Distribution of this document is unlimited.~~

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

The object of this study is to find an approximation to the discrete optimal Kalman filter gain schedule by closed-form analytic expressions. In doing so, required table storage and/or on-line computation time can be reduced at little expense in terms of filter performance degradation. The method of least squares was used to determine the closed-form solution which was the best fit to the discrete Kalman filter gain schedule. The criterion for performance degradation was the difference between the values of the diagonal elements of the estimation covariance matrix, $P_{k/k}$, obtained by using the Kalman gain schedule, and the corresponding values obtained by using the closed-form analytic expressions for the elements of the gain matrix. Examples are presented to show that near-optimal results were obtained utilizing this method. A comparison of the results of this study with another near-optimal estimation scheme is also included.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Discrete Kalman Filtering						
Sub-Optimal Estimation						
Least-Squares Approximation						
Closed-Form Filter Gain Schedule						